ESC103 Unit 19

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1 Boundary Value Problems

Suppose we want to solve the following differential equation:

$$y'' + y' + y = 0$$

on the interval

$$[0, 1]$$

and we are given information about y at the end of the interval [0,1]:

$$y(0) = 0, y(1) = 1$$

It's different from **IVPs:**

$$y'' + 2y' + y = 0$$

$$[0, 1] \rightarrow y(0) = 0, y'(1) = 1$$

But in this case we are given information about y ONLY on the left side of the interval.

Another important difference between IVPs and BVPs is that the independent variable associated with IVPs is typically time, but with BVPs the independent variable is typically a spatial variable (x).

A numerical approach to solving BVPs begins by partitioning the interval [a,b] into n evenly spaced sub-intervals.

$$a = \dot{x_0} \;\; \dot{x_1} \;\; \dot{x_2} \;\; \dot{x_{n-1}}$$

$$\therefore \text{Step size: } \Delta x = \frac{b-a}{n}$$

Then the derivatives in the D.E. are approximated sing finite differences. This step converts the differential equation into a difference equation. The difference equation will be written at each interior grid point to produce a system of linear algebraic equations of the form $A\overrightarrow{x} = \overrightarrow{b}$

$$\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

Different finite differences are available.

Forward difference:

$$\Delta_F f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Backward difference:

$$\Delta_B f(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Central Difference:;

$$\Delta_C f(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

We can use different combinations of these finite differences may be used to approximate higher derivatives.

$$f''(x) \approx \Delta_B[\Delta_F f(x)]$$

$$= \frac{\Delta_F f(x) - \Delta_F f(x-\Delta x)}{\Delta x}$$

$$= \frac{1}{\Delta x}\left[\frac{f(x+\Delta x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x}\right]$$

Simplifying:

$$= \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

Example: Given:

$$y'' + 2y' + y = 0 \qquad [0, 1]$$

and:

$$y(0) = 0, \quad y(1) = 1$$

Here we will choose:

$$y'(x) \approx \Delta_c y(x)$$

$$y''(x) \approx \Delta_B \Delta_F y(x)$$

$$\frac{y(x + \Delta x) - 2y(x) + y(x - \Delta x)}{(\Delta x)^2} + 2\left(\frac{y(x + \Delta x) - y(x - \Delta x)}{2\Delta x}\right) + y(x) =$$

$$\left(\frac{1}{(\Delta x)^2} + \frac{2}{2\Delta x}\right) y(x+\Delta x) + \left(\frac{-2}{(\Delta x)^2} + 1\right) y(x) + \left(\frac{1}{(\Delta x)^2} - \frac{2}{2\Delta x}\right) y(x-\Delta x) = 0$$

Let's decide to use 5 sub-intervals (n = 5) to divide up [0, 1]

$$\Delta x = \frac{1 - 0}{5} = 0.2$$

$$x \doteq 0 \quad x \doteq 2 \quad x \doteq 6 \quad x \doteq 8$$

$$30y(x + \Delta x) - 49y(x) + 20y(x - \Delta x) = 0$$

if

$$x = 0$$

we are given by the boundary condition the value if

$$x = 0.2$$

then $x + \Delta x$ is 0.4, therefore:

$$30y(0.4) - 49y(0.2) + 20y(0) = 0$$

Next inner grid point:

$$x = 0.4$$

$$30y(0.6) - 49y(0.4) + 20y(0.2) = 0$$

Next:

$$x = 0.6$$

$$30y(0.8) - 49y(0.6) + 20y(0.4 = 0)$$

Next:

$$x = 0.8$$

$$30y(1) - 49y(0.8 + 20y(0.6) = 0$$

We know the values at the end points, so we don't need to write those explicitly as equations.

Formulate as $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} -49 & 30 & 0 & 0 \\ 20 & -49 & 30 & 0 \\ 0 & 20 & -49 & 30 \\ 0 & 0 & 20 & -49 \end{bmatrix} \begin{bmatrix} y(0.2) \\ y(0.4) \\ y(0.6) \\ y(0.8) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -30 \end{bmatrix}$$

... Gaussian Elimination

$$\overrightarrow{x} = \begin{bmatrix} y(0.2) \\ y(0.4) \\ y(0.6) \\ y(0.8) \end{bmatrix} = \begin{bmatrix} 0.4493 \\ 0.7338 \\ 0.8990 \\ 0.9792 \end{bmatrix}$$